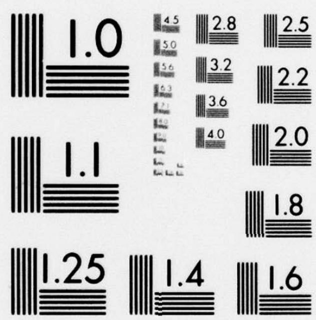
AD-A073 896 MISSOURI UNIV-COLUMBIA DEPT OF STATISTICS F/G 12/1
CONFIDENCE INTERVALS FOR AN EXPONENTIAL PARAMETER FROM A TWO ST--ETC(U)
AUG 79 K B FAIRBANKS N00014-76-C-0789
UNCLASSIFIED TR-87 NL

| OF |
AD
A073896

END
DATE
FILMED
10-79
DDC



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL II

8

AD A073896

University of Missouri-Columbia

Confidence Intervals for an Exponential Parameter from a Two Stage Life Test

by

Kenneth B. Fairbanks

Technical Report No. 87
Department of Statistics

August 1979

Mathematical Sciences

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
87

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
6 Confidence Intervals for an Exponential Parameter from a Two Stage Life Test

5. TYPE OF REPORT & PERIOD COVERED
9 Technical Report

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
10 Kenneth B. Fairbanks

8. CONTRACT OR GRANT NUMBER(s)
15 N00014-76-C-0789

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics
University of Missouri - Columbia
Columbia, MO 65211

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NR 042-353 12 34p.

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Department of the Navy
Arlington, VA 22217

12. REPORT DATE
11 August 1979

13. NUMBER OF PAGES
30

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
14 TR-87

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Confidence interval, two stage test, exponential distribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The two stage life test of Bulgren and Hewett (1973) for the mean lifetime, θ, of an exponentially distributed lifetime is essentially a two stage version of type II censoring. Since the test decision only determines whether θ is above or below θ_0, it may be of interest to estimate θ following the decision using the test data. Epstein (1960a) found confidence intervals for under type II censoring. This report shows that Epstein's intervals may be modified slightly

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

JOB

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

402 660

20.

and used to provide confidence intervals for θ following the two stage test. The resulting confidence intervals are shown to be conservative.

Confidence Intervals for an Exponential Parameter

from a Two Stage Life Test

by

Kenneth B. Fairbanks*

University of Missouri - Columbia

Technical Report No. 87

August 1979

Prepared under contract N00014-76-C-0789

(NR-042-353)

Office of Naval Research

Richard W. Madsen, Project Director

This report is based on part of the author's Ph.D. dissertation.

University of Missouri

Department of Statistics

Columbia, Missouri 65211

Confidence Intervals for an Exponential

Parameter from a Two Stage Life Test

Abstract

Theta

The two stage life test of Bulgren and Hewett (1973) for the mean lifetime, θ, of an exponentially distributed lifetime is essentially a two stage version of type II censoring. Since the test decision only determines whether θ is above or below θ_0, it may be of interest to estimate θ following the decision using the test data. Epstein (1960a) found confidence intervals for under type II censoring. This report shows that Epstein's intervals may be modified slightly and used to provide confidence intervals for θ following the two stage test. The resulting confidence intervals are shown to be conservative.

Theta sub ∅

Theta

Confidence Intervals for an Exponential
Parameter from a Two Stage Life Test

I. Introduction.

In the theory of reliability and life testing, a lifetime τ is often assumed to have an exponential distribution. Thus the probability density function of τ may be expressed as

$$f(\tau;\theta) = \begin{cases} \frac{1}{\theta} e^{-\tau/\theta} & \text{if } \tau \geq 0 \\ 0 & \text{elsewhere} \end{cases} \qquad (1.1)$$

where $\theta > 0$ is the expected lifetime.

In a life test with type II censoring, n items are placed on test and observed for failures until the r_0^{th} failure occurs. Properties of this test were given by Epstein and Sobel (1953). Bulgren and Hewett (1973) developed a two stage version of a life test with type II censoring to test $H_0: \theta \geq \theta_0$ vs. $H_1: \theta < \theta_0$. In the test n items are placed on test and two integer values, r_1 and r_2 are chosen such that $r_1 + r_2 \leq n$. In the first stage, testing proceeds (with or without the replacement of failed items) until r_1 failures occur. At this time a decision is made to accept or reject H_0, or to continue on to the second stage where r_2 additional failures are

observed and a final decision is made. The decisions are based on the values of T_{r_1} and T_{r_3}, the total times on test at the times of the $r_1{}^{th}$ and $r_3{}^{th}$ failures repectively, with $r_3 = r_1 + r_2$. The test is depicted in figure 1.1.

It is possible that additional information about the true value of θ may be needed following the test decision. Thus it may be of interest to estimate θ using the data from the life test. In this report we derive confidence intervals for θ using data from the two stage life test.

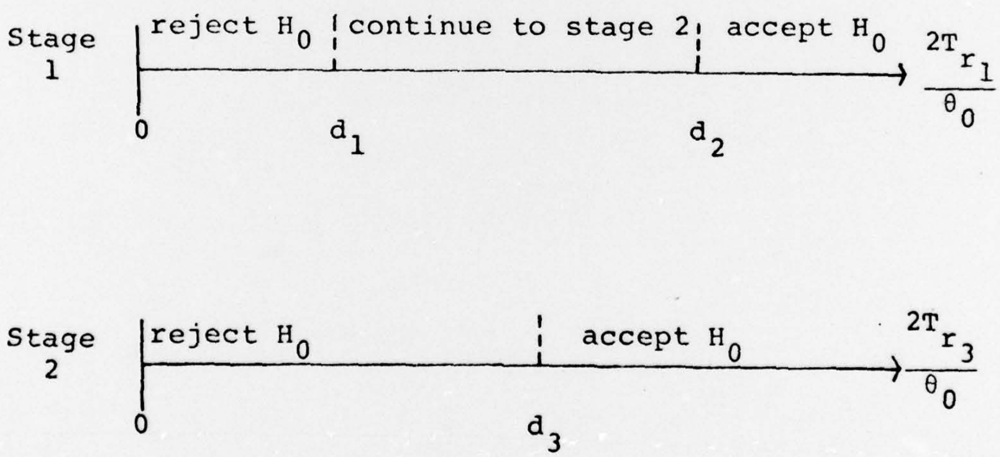


Figure 1.1. The two stage life test.

II. A Two Sided Confidence Interval for θ

Confidence intervals for θ using type II censoring were given by Epstein (1960a) and Epstein (1960b). A two

sided interval with confidence coefficient $(1 - \alpha)$ is

$$\left[\frac{2T_r}{\chi^2_{\frac{\alpha}{2},2r}} \; , \; \frac{2T_r}{\chi^2_{1-\frac{\alpha}{2},2r}} \right] \tag{2.1}$$

where T_r is the total time on test at the r^{th} failure. If $t_1, t_2, \ldots, t_n$ represent the ordered failure times, then

$$T_r = \begin{cases} \sum_{i=1}^{r} t_i + (n - r)t_r & \text{(without replacement)} \\ nt_r & \text{(with replacement)} \end{cases} .$$

A two sided confidence interval for type II censoring in a single stage is given by (2.1). This suggests the possibility that the single stage interval could be used at either stage of the two-stage test. More formally, we conjecture that the following rule provides a $100(1-\alpha')$ percent confidence interval for θ.

When the decision is made at stage 1, use

$$\left[\frac{2T_{r_1}}{\chi^2_{\frac{\alpha'}{2},2r_1}} \; , \; \frac{2T_{r_1}}{\chi^2_{1-\frac{\alpha'}{2},2r_1}} \right] .$$

When the decision is made at stage 2, use (2.2)

$$\left[\frac{2T_{r_3}}{\chi^2_{\frac{\alpha'}{2},2r_3}}\,,\,\frac{2T_{r_3}}{\chi^2_{1-\frac{\alpha'}{2},2r_3}}\right].$$

We shall refer to the first interval in (2.2) as I_1 and the second interval as I_2. In general we will refer to the single interval generated by rule (2.2) as I. To establish that I is a confidence interval for θ with a coefficient at least $1-\alpha'$, we would have to show that $P(\theta\varepsilon I) \geq 1 - \alpha'$. Unfortunately, it will be shown that $P(\theta\varepsilon I) < 1 - \alpha'$ for some θ. However, we will also show that, for a suitably chosen α', the confidence interval given by (2.1) will have a confidence coefficient no less than $1 - \alpha$.

First, define event $D \equiv$ (the test decision is made at stage 1). Then $\bar{D}$ will represent continuation to stage 2. We may then write

$$P(\theta\varepsilon I) = P(D,\theta\varepsilon I_1) + P(\bar{D},\theta\varepsilon I_2) = p_1 + p_2, \text{ say.}$$

If we recall that $2T_{r_1}/\theta \sim \chi^2_{2r_1}$, it is seen that $P(\theta\varepsilon I_1) = 1 - \alpha'$. Also, $P(\theta\varepsilon I_1) = P(D,\theta\varepsilon I_1) + P(\bar{D},\theta\varepsilon I_1) = P_1 + P(\bar{D},\theta\varepsilon I_1)$. Combining these results gives

$$P(\theta \varepsilon I) = 1 - \alpha' - (P(\bar{D}, \theta \varepsilon I_1) - P(\bar{D}, \theta \varepsilon I_2))$$

$$= 1 - \alpha' - p^*, \text{ say.}$$

To establish the validity of the interval in (2.1) it would be necessary that $p^* = P(\bar{D}, \theta \varepsilon I_1) - P(\bar{D}, \theta \varepsilon I_2) \leq 0$. That this is not always the case will be shown later. We first introduce some notational definitions which should make subsequent material more readable.

Define the transformations $Y = 2T_{r_1}/\theta$ and $Z = 2T_{r_3}/\theta$ so that $Y \sim \chi^2_{2r_1}$ and $Z \sim \chi^2_{2r_3}$. Under these transformations, p^* is equal to

$$P\left[\frac{\theta_0 d_1}{\theta} \leq Y \leq \frac{\theta_0 d_2}{\theta},\ \chi^2_{1-\frac{\alpha'}{2},2r_1} \leq Y \leq \chi^2_{\frac{\alpha'}{2},2r_1}\right] -$$

$$P\left[\frac{\theta_0 d_1}{\theta} \leq Y \leq \frac{\theta_0 d_2}{\theta},\ \chi^2_{1-\frac{\alpha'}{2},2r_3} \leq Z \leq \chi^2_{\frac{\alpha'}{2},2r_3}\right].$$

Now define

$$R_1 = \left[\chi^2_{1-\frac{\alpha'}{2},2r_1},\ \chi^2_{\frac{\alpha'}{2},2r_1}\right]$$

$$R_2 = \left[\frac{\theta_0 d_1}{\theta},\ \frac{\theta_0 d_2}{\theta}\right], \text{ and}$$

$$R_3 = \left[\chi^2_{1-\frac{\alpha'}{2},2r_3},\ \chi^2_{\frac{\alpha'}{2},2r_3}\right]$$

Thus, under the transformations, R_1, R_2 and R_3 correspond to I_1, the continuation region, and I_3 respectively. This permits us to write p* simply as $P(Y\varepsilon R_2, Y\varepsilon R_1)$ - $P(Y\varepsilon R_2, Z\varepsilon R_3)$.

Making use of the endpoints of intervals R_1, R_2 and R_3, we can make some general observations about $P(\theta\varepsilon I)$. We consider those intervals designated by numbers in parentheses in Figure 2.1.

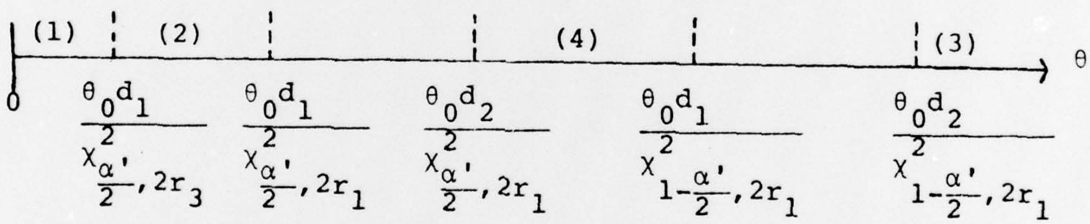


Figure 2.1. A partition of the parameter space.

In interval (1):

$$\theta < \frac{\theta_0 d_1}{\chi^2_{\frac{\alpha'}{2},2r_3}}$$

In this interval,

$$\frac{\theta_0 d_1}{\theta} > \chi^2_{\frac{\alpha'}{2},2r_3} > \chi^2_{\frac{\alpha'}{2},2r_1} ,$$

which implies that $P(Y\varepsilon R_2, Z\varepsilon R_3) = P(Y\varepsilon R_2, Y\varepsilon R_1) = 0$. Thus, in interval (1) we have $P(\theta\varepsilon I) = 1 - \alpha'$.

In interval (2):

$$\frac{\theta_0 d_1}{\chi^2_{\frac{\alpha'}{2},2r_3}} \leq \theta < \frac{\theta_0 d_1}{\chi^2_{\frac{\alpha'}{2},2r_1}}$$

Since

$$\frac{\theta_0 d_1}{\theta} > \chi^2_{\frac{\alpha'}{2},2r_1}$$

in this interval, we have $P(Y\varepsilon R_2, Y\varepsilon R_1) = 0$. This gives the result that $P(\theta\varepsilon I) \geq 1 - \alpha'$ for all θ in interval (2).

In interval (3):

$$\theta \geq \frac{\theta_0 d_2}{\chi^2_{1-\frac{\alpha'}{2},2r_1}}$$

Here

$$\frac{\theta_0 d_2}{\theta} \leq \chi^2_{1-\frac{\alpha'}{2},2r_1}$$

implies $P(Y\varepsilon R_2, Y\varepsilon R_1) = 0$. So again, for θ in the interval (3), $P(\theta\varepsilon I) \geq 1 - \alpha'$.

In interval (4):

$$\frac{\theta_0 d_2}{\chi^2_{\frac{\alpha'}{2},2r_1}} \leq \theta \leq \frac{\theta_0 d_1}{\chi^2_{1-\frac{\alpha'}{2},2r_1}}$$

We note that, if

$$(d_2/d_1) \geq (\chi^2_{\frac{\alpha'}{2},2r_1} / \chi^2_{1-\frac{\alpha'}{2},2r_1}),$$

the sense of the inequalities will be reversed and interval (4) will be empty. When it is non-empty we have $R_2 \subseteq R_1$ and

$$\begin{aligned} P(\theta\varepsilon I) &= 1-\alpha'-P(Y\varepsilon R_2) + P(Y\varepsilon R_2, Z\varepsilon R_3) \\ &= 1-\alpha'-P(Y\varepsilon R_2)(1-P(Z\varepsilon R_3 | Y\varepsilon R_2)) \\ &= 1-\alpha'-P(\bar{D})(1-P(\theta\varepsilon I_2 | \bar{D})). \end{aligned}$$

Since $P(\bar{D}) > 0$ and $P(\theta\varepsilon I_2|\bar{D}) < 1$, it follows that $P(\theta\varepsilon I) < 1 - \alpha'$ for θ in interval (4) when it is non-empty. This shows that the confidence interval proposed by (2.1) is invalid in the sense that, for some θ, $P(\theta\varepsilon I) < 1 - \alpha'$.

For those regions not included in these four intervals a general pattern of the behavior of $P(\theta\varepsilon I)$ as a function of θ was observed by computing $P(\theta\varepsilon I)$ on the computer for various test parameters given by Bulgren and

Hewett (1973). The general pattern which was observed is depicted in Figure 2.2. It can be seen from Figure 2.2 that if α' were chosen sufficiently small, it would be possible to keep $P(\theta \varepsilon I) \geq 1 - \alpha$, where $1 - \alpha$ is the desired confidence coefficient. An appropriate value of α' can be found if we find an upper bound on p^* which can be solved for α'. Recall that p^* is just $P(Y\varepsilon R_2, Y\varepsilon R_1) - P(Y\varepsilon R_2, Z\varepsilon R_3)$. The difficulty in finding a bound on p^* is finding a closed expression for $P(Y\varepsilon R_2, Z\varepsilon R_3)$ which is also simple enough to permit the solution for α'. Bulgren and Hewett (1973) give the joint density of Y and Z as

$$f(y,z) = \begin{cases} \dfrac{(z-y)^{r_2-1} y^{r_1-1} e^{-z/2}}{2^{r_1+r_2}\Gamma(r_1)\Gamma(r_2)} & 0<y<z \quad 0<z<\infty \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$P(Y\varepsilon R_2, Z\varepsilon R_3) = \int_{\frac{\theta_0 d_1}{\theta}}^{\frac{\theta_0 d_2}{\theta}} \int_{\chi^2_{1-\frac{\alpha'}{2},2r_3}}^{\chi^2_{\frac{\alpha'}{2},2r_3}} f(y,z)\,dzdy,$$

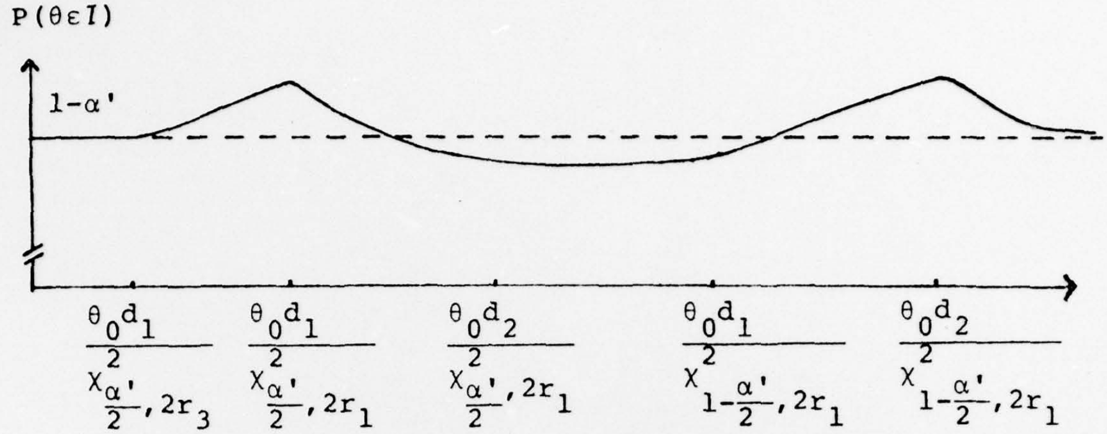


Figure 2.2. Behavior of $P(\theta \varepsilon I)$ when I is from (2.1)

and it is apparent that the closed, simple expression we seek is not possible. In lieu of evaluating this joint probability directly, we will simplify it using a Bonferroni bound. A lower bound on $P(Y \varepsilon R_2, Z \varepsilon R_3)$, using the Bonferroni inequality, is

$$\begin{aligned} P(Y\varepsilon R_2, Z\varepsilon R_3) &\geq P(Y\varepsilon R_2) + P(Z\varepsilon R_3) - 1 \\ &= P(Y\varepsilon R_2) + (1-\alpha') - 1 \\ &= P(Y\varepsilon R_2) - \alpha'. \end{aligned}$$

Thus

$$p^* \leq P(Y\varepsilon R_1, Y\varepsilon R_2) - P(Y\varepsilon R_2) + \alpha'. \qquad (2.3)$$

This upper bound on p* may be further simplified with some observations about the relationship of R_1 to R_2. With respect to θ, interval R_1 is fixed. On the other hand, R_2 increases in both size and location as θ decreases. When $\theta = \infty$, R_2 is the single point $\{0\}$. When θ is such that $\theta_0 d_1/\theta$ coincides with

$$2T_{r_1}/\chi^2_{1-\frac{\alpha'}{2}, 2r_1},$$

it could be that $R_2 \subseteq R_1$. If this occurs, the upper bound in (2.3) achieves its maximum value α'. Since $P(\theta \varepsilon I) = 1 - \alpha' - p^*$, the Bonferroni inequality gives $P(\theta \varepsilon I) \geq 1 - 2\alpha'$. Obviously, if we want $P(\theta \varepsilon I) \geq 1 - \alpha$, α' is given the value $\alpha/2$. This establishes the following result.

Theorem 2.1. Let r_1 and r_2 be the number of failures required for each stage of the two-stage exponential life test, where $r_3 = r_1 + r_2$. If T_{r_1} and T_{r_3} represent the total time on test at the r_1th and r_3th failures respectively, then the following rule gives a two-sided confidence interval for the exponential mean θ, with a confidence coefficient at least $1 - \alpha$, at the time of decision for the two-stage test. If a decision is made at the first stage, use

$$\left[\frac{2T_{r_1}}{\chi^2_{\frac{\alpha}{4},2r_1}}, \frac{2T_{r_1}}{\chi^2_{1-\frac{\alpha}{4},2r_1}}\right].$$

If a decision is made at the second stage, use

$$\left[\frac{2T_{r_3}}{\chi^2_{\frac{\alpha}{4},2r_3}}, \frac{2T_{r_3}}{\chi^2_{1-\frac{\alpha}{4},2r_3}}\right].$$

The wide applicability of the Bonferroni inequality suggests that the bound in (2.3) may be quite crude, leading to overly conservative confidence intervals. An alternative approach will be employed in searching for a less conservative confidence interval. We also will admit the possibility that $P(\theta\varepsilon I) < 1 - \alpha$, for some θ, if the difference $(1-\alpha)-P(\theta\varepsilon I)$ can be shown to be small. This would result in approximate confidence intervals.

Let

$$\Delta = P(Y\varepsilon R_2)P(Z\varepsilon R_3) - P(Y\varepsilon R_2, Z\varepsilon R_3).$$

Then

$$\begin{aligned} p^* &= P(Y\varepsilon R_1, Y\varepsilon R_2) - P(Y\varepsilon R_2)P(Z\varepsilon R_3) + \Delta \\ &= P(Y\varepsilon R_1, Y\varepsilon R_2) - P(Y\varepsilon R_2)(1-\alpha') + \Delta. \qquad (2.4) \end{aligned}$$

If $\Delta = 0$ (this would be true if Y and Z were independent) p^* could be bounded above without use of the Bonferroni inequality. The penalty for assuming $\Delta = 0$, when, in fact, $\Delta \neq 0$ could be a confidence interval where $P(\theta \varepsilon I) < 1 - \alpha$ for some θ. The difference $(1-\alpha) - P(\theta \varepsilon I)$ would be at most Δ, and the Bonferroni inequality can provide an upper bound on Δ.

$$P(Y\varepsilon R_1, Z\varepsilon R_3) \geq P(Y\varepsilon R_2) + P(Z\varepsilon R_3) - 1$$
$$= P(Y\varepsilon R_2) - \alpha'.$$

Thus

$$\Delta \leq P(Y\varepsilon R_2)(1-\alpha') - P(Y\varepsilon R_2) + \alpha'$$
$$= \alpha'(1-P(Y\varepsilon R_2))$$
$$= \alpha' P(\text{Decision at stage 1}). \qquad (2.5)$$

Since this Bonferroni bound may also be overly conservative, and since α' will be less than α, it is apparent that Δ will be small if it is positive. If Δ is negative the resulting confidence interval would be less conservative than under the assumption $\Delta = 0$. Thus, if we assume $\Delta = 0$ and then bound p^* under this assumption, the resulting confidence interval will be such that $P(\theta \varepsilon I) \geq 1 - \alpha - \Delta$, where Δ is bounded above by (2.5). We proceed with the assumption that $\Delta = 0$.

The expression (2.4) for p^* now becomes

$$p^* = P(Y\varepsilon R_1, Y\varepsilon R_2) - P(Y\varepsilon R_2)(1-\alpha'). \qquad (2.6)$$

To establish an upper bound on (2.6) we again consider the relationship of R_1 with R_2 as θ changes. Our observations will pivot on the relationship of R_1 with R_2 when θ is such that

$$\theta_0 d_1/\theta = \chi^2_{1-\frac{\alpha'}{2},2r_1},$$

i.e., when the left endpoints of R_1 and R_2 coincide. Here either $R_2 \subseteq R_1$ or $R_1 \subseteq R_2$. We will designate the situation $R_2 \subseteq R_1$ as *Case I, and Case II will correspond to* $R_1 \subseteq R_2$. Case I and Case II are depicted in Figure 2.3 . In a specific application, whether Case I or Case II applies will depend on the test parameters involved and the value of α'. Each case is considered separately.

Case I: $R_2 \subseteq R_1$ when their left endpoints coincide. We examine the nature of the relationship of R_1 with R_2 for different θ and its effect on the problem of placing a bound on p^*.

I.a) θ is such that $R_1 \cap R_2 = \emptyset$.

Referring to (2.6) it is clear that $p^* \leq 0$ for these values of θ.

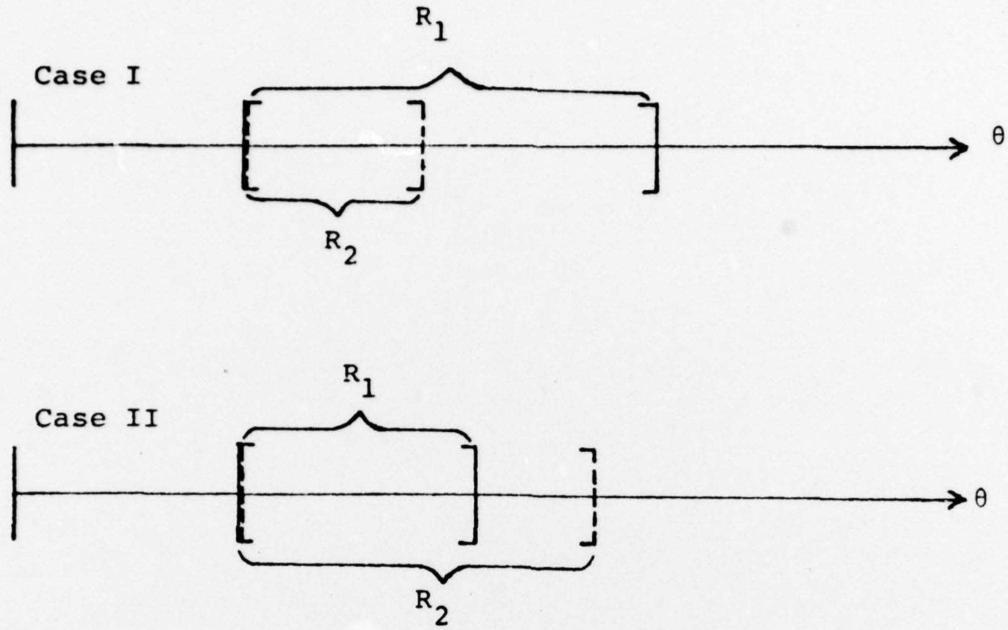


Figure 2.3. Relationship of R_1 with R_2 when θ is such that their left endpoints coincide.

I.b) θ is such that $R_1 \cap R_2 \neq \emptyset$ and $\bar{R}_1 \cap R_2 \neq \emptyset$. Let $R_1 \cap R_2 \equiv R_5$ and $\bar{R}_1 \cap R_2 \equiv R_4$. With this notation we have

$$\begin{aligned} p^* &= P(Y\varepsilon R_1, Y\varepsilon R_2) - (1-\alpha')P(Y\varepsilon R_2) \\ &= P(Y\varepsilon R_5) - (1-\alpha')[P(Y\varepsilon R_4) + P(Y\varepsilon R_5)] \\ &= \alpha' P(Y\varepsilon R_5) - (1-\alpha')P(Y\varepsilon R_4). \qquad (2.7) \end{aligned}$$

Expression (2.7) is maximum when $R_4 = \emptyset$ or, equivalently $R_2 \subseteq R_1$.

It is clear that, to establish an upper bound on p^* for Case I, it suffices to consider only those θ for which $R_2 \subseteq R_1$. If $R_2 \subseteq R_1$ expression (2.6) reduces to $p^* = \alpha' P(Y \varepsilon R_2)$. Note first that if $R_2 \subseteq R_1$, then $P(Y \varepsilon R) \leq P(Y \varepsilon R_1) = 1 - \alpha'$. So, one bound on p^* under Case I is $\alpha'(1-\alpha')$. A smaller bound may be found if we determine the maximum value of $P(Y \varepsilon R_2)$, subject to $R_2 \subseteq R_1$.

For that purpose, we find the value of θ, say θ^*, for which $P(Y \varepsilon R_2)$ is maximum for given θ_0, d_1 and d_2. Since $Y \sim \chi^2_{2r_1}$ we can write

$$P(Y \varepsilon R_2 | \theta) = \int_{\ell(\theta)}^{u(\theta)} f(t)\,dt, \text{ where } \ell(\theta) = \theta_0 d_1/\theta,$$

$$u(\theta) = \theta_0 d_2/\theta, \text{ and } f(t) = \frac{t^{r_1 - 1} e^{-t/2}}{2^{r_1} \Gamma(r_1)} .$$

Then

$$\frac{\partial P(Y \varepsilon R_2 | \theta)}{\partial \theta} = f(u(\theta)) \frac{\partial u(\theta)}{\partial \theta} - f(\ell(\theta)) \frac{\partial \ell(\theta)}{\partial \theta} .$$

Setting

$$\frac{\partial P(Y \varepsilon R_2 | \theta)}{\partial \theta} = 0$$

and making the appropriate substitutions, we obtain

$$d_2\left(\frac{\theta_0 d_2}{\theta}\right)^{r_1-1} e^{-\frac{\theta_0 d_2}{2\theta}} = d_1\left(\frac{\theta_0 d_1}{\theta}\right)^{r_1-1} e^{-\frac{\theta_0 d_1}{2\theta}},$$

or

$$\left(\frac{d_2}{d_1}\right)^{r_1} e^{\frac{\theta_0}{2\theta}(d_1-d_2)} = 1.$$

Finally

$$\theta^* = \frac{\theta_0(d_2-d_1)}{2r_1(\ln d_2 - \ln d_1)}.$$

So, $P(Y\varepsilon R_2|\partial)$ increases to a maximum at θ^* and then decreases. Now let

$$R_2^* = \left[\frac{\theta_0 d_1}{\theta^*}, \frac{\theta_0 d_2}{\theta^*}\right].$$

If $R_2^* \subseteq R_1$, an upper bound on p^* is clearly $\alpha' P(Y\varepsilon R_2^*)$ under Case I.

Since

$$R_1 = \left[\chi^2_{1-\frac{\alpha'}{2},2r_1}, \chi^2_{\frac{\alpha'}{2},2r_1}\right]$$

and α' is as yet unknown, it appears that it is not possible to determine if $R_2^* \subseteq R_1$. We note however that $\alpha' \leq \alpha$ and consequently

$$R_2^* \subseteq \left[\chi^2_{1-\frac{\alpha}{2},2r_1}, \chi^2_{\frac{\alpha}{2},2r_1}\right]$$

implies that

$$R_2^* \subseteq \left[\chi^2_{1-\frac{\alpha'}{2},2r_1}, \chi^2_{\frac{\alpha'}{2},2r_1}\right].$$

Thus in most cases, using the desired $1 - \alpha$ coefficient, it can be determined if $R_2^* \subseteq R_1$. When this relationship exists we find that $P(\theta \varepsilon I) \geq 1 - \alpha' - \alpha' P(Y \varepsilon R_2^*)$. So, if $\alpha' = \alpha/[1+P(Y\varepsilon R_2^*)]$, we have $P(\theta \varepsilon I) \geq 1 - \alpha$. We also note that

$$\frac{\alpha}{1 + P(Y\varepsilon R_2^*)} > \frac{\alpha}{2}$$

insures that the interval with $\alpha' = \alpha/[1+P(Y\varepsilon R_2^*)]$ will be less conservative than the original intervals in Theorem 2.1 using $\alpha' = \alpha/2$.

Computations using actual test parameters given by Bulgren and Hewett (1973) suggest that R_2^* is quite commonly contained in R_1 for smaller values of α, say

$\alpha \leq .2$. The improvement over the intervals of Theorem 2.1 will be discussed later in this section.

If $R_2^* \nsubseteq R_1$, the problem becomes more complex. Here $P(Y \varepsilon R_2 | \theta)$, subject to the constraint $R_2 \subseteq R_1$, reaches its maximum value when the left endpoints of R_1 and R_2 coincide, or when the right endpoints coincide, depending on the location of R_2^* with respect to R_1. Specifically, when $R_1 \supseteq R_2$,

$$\max P(Y \varepsilon R_2 | \theta) = \begin{cases} \displaystyle\int_{\chi^2_{1-\frac{\alpha'}{2}, 2r_1}}^{\frac{d_2}{d_1} \chi^2_{1-\frac{\alpha'}{2}, 2r_1}} f(t)\,dt, & \text{if } \dfrac{\theta_0 d_1}{\theta^*} < \chi^2_{1-\frac{\alpha'}{2}, 2r_1} \\[2ex] \displaystyle\int_{\frac{d_1}{d_2} \chi^2_{\frac{\alpha'}{2}, 2r_1}}^{\chi^2_{\frac{\alpha'}{2}, 2r_1}} f(t)\,dt, & \text{if } \dfrac{\theta_0 d_1}{\theta^*} \geq \chi^2_{1-\frac{\alpha'}{2}, 2r_1} \end{cases}$$

We need to solve the equation

$$\alpha = \alpha' + \alpha'(\max P(Y \varepsilon R_2 | \theta))$$

for α'. The difficulty involved is the manner in which α' appears in the limits of integration in max $P(Y\varepsilon R_2|\theta)$. Consequently, if it is found that $R_2^* \subseteq R_1$ cannot be established, the use of $\alpha'(1-\alpha')$ as an upper bound on p^* is advised. We next treat the corresponding problem under Case II.

Case II: When θ is such that the left endpoints of R_1 and R_2 coincide, $R_1 \subseteq R_2$. By arguments essentially the same as those used for Case I, it suffices to consider only the set of θ for which $R_1 \subseteq R_2$, in finding an upper bound on p^*.

If $R_1 \subseteq R_2$,

$$\begin{aligned} p^* &= P(Y\varepsilon R_2, Y\varepsilon R_1) - (1-\alpha')P(Y\varepsilon R_2) \\ &= P(Y\varepsilon R_1) - (1-\alpha')P(Y\varepsilon R_2) \\ &= (1-\alpha') - (1-\alpha')P(Y\varepsilon R_2) \\ &= (1-\alpha')[1-P(Y\varepsilon R_2)]. \end{aligned}$$

Now, since $P(Y\varepsilon R_2) \geq P(Y\varepsilon R_1) = 1 - \alpha'$, we again find, as we did in Case I, that $p^* \leq \alpha'(1-\alpha')$. This bound is achieved by p^* if $R_1 = R_2$.

We have shown that, under the assumption $\Delta = 0$, an upper bound on p^* is $\alpha'(1-\alpha')$, and $P(\theta\varepsilon I) \geq 1 - \alpha' - \alpha'(1-\alpha')$. If we want $P(\theta\varepsilon I) \geq 1 - \alpha$, then set $\alpha' = 1 - (1-\alpha)^{1/2}$. In Theorem 2.1 we used $\alpha' = \alpha/2$ which is

greater than $1 - (1-\alpha)^{1/2}$. However, for small α, $\alpha/2 \approx 1 - (1-\alpha)^{1/2}$. Thus, the improvement over the results in Theorem 2.1 is slight. A further improvement is possible when $R_2^* \subseteq R_1$. Then we have $\alpha' = \alpha/[1+P(Y\varepsilon R_2^*)]$. The degree of improvement in this situation over the intervals of Theorem 2.1 depends on the value of $P(Y\varepsilon R_2^*)$.

To get some idea of how conservative the interval in Theorem 2.1 is, some computer studies were carried out using specific two-stage test parameters given by Bulgren and Hewett (1973). For each test used, the interval

$$\left[\frac{\theta_0 d_1}{\chi^2_{\frac{\alpha'}{2},2r_1}}, \frac{\theta_0 d_2}{\chi^2_{1-\frac{\alpha'}{2},2r_1}}\right]$$

(see Figure 2.2) was divided by twenty points. Using $\alpha' = \frac{\alpha}{2}$, $P(\theta\varepsilon I)$ was computed at each of the twenty points. Numerical integration techniques were used to find $P(Y\varepsilon R_2, Z\varepsilon R_3)$. Table 2.1 gives the minimum and maximum values of $P(\theta\varepsilon I)$ over the twenty points. Many other tests were included in this study but, since results on those were nearly the same as for the tests given in Table 2.1 , their results have not been included in the Table.

Table 2.2 reflects the improvement which is possible if it happens that $R_2^* \subseteq R_1$ and the value used for α' is $\alpha/1+P(Y\varepsilon R_2^*)$ rather than $1-(1-\alpha)^{1/2}$. The desired confidence coefficent in Table 2.2 is .90. As in Table 2.1 the minimum and maximum observed values of $P(\theta\varepsilon I)$ are given, first using $\alpha' = 1-(1-\alpha)^{1/2}$, and then using $\alpha' = \alpha/1+P(Y\varepsilon R_2^*)$. The confidence intervals using the latter expression for α' are less conservative, but the improvement is hardly dramatic. We do not sacrifice much by using the simpler intervals of Theorem 2.1.

III. A One-Sided Confidence Interval

If a one-sided confidence interval for θ is desired following the two-stage test, the same approach as used in Section II will yield analogous results. The confidence interval is found according to the following rule.

When the test decision is made at stage one, use

$$\left[\frac{2T_{r_1}}{\chi^2_{\alpha',2r_1}}, \infty\right) = I_1, \text{ say.}$$

When the test decision comes at stage two, use (3.1)

$$\left[\frac{2T_{r_3}}{\chi^2_{\alpha',2r_3}}, \infty\right) = I_2, \text{ say.}$$

TABLE 2.1

Extreme Values of $P(\theta \varepsilon I)$ Using Theorem 2.1

Test Parameters			Nominal Confidence Coefficients			
θ_0/θ_1	α	β	.99	.95	.90	.80
2	.05	.05	.9973	.9855	.9692	.9328
			.9931	.9650	.9285	.8497
3	.05	.05	.9973	.9856	.9693	.9328
			.9931	.9648	.9278	.8466
4	.05	.05	.9974	.9863	.9710	.9365
			.9926	.9600	.9147	.8327

Note: The first entry in each cell is the maximum observed $P(\theta \varepsilon I)$ over the twenty points used.

The second entry is the minimum observed $P(\theta \varepsilon I)$.

TABLE 2.2

Extreme Values of $P(\theta \varepsilon I)$ Using Two Methods

Test Parameters θ_0/θ_1	α	β	α'	Minimum $P(\theta\varepsilon I)$	Maximum $P(\theta\varepsilon I)$
$\frac{5}{2}$	.05	.05	.0513	.9272	.9698
			.0592	.9143	.9642
3	.05	.05	.0513	.9278	.9636
			.0595	.9152	.9690
2	.10	.10	.0513	.9290	.9690
			.0608	.9166	.9634
$\frac{5}{2}$	.10	.10	.0513	.9257	.9701
			.0583	.9125	.9646
3	.10	.10	.0513	.9241	.9702
			.0566	.9105	.9647

Note: The first entry in a cell corresponds to $\alpha' = 1 - (1-\alpha)^{1/2}$.

The second entry corresponds to $\alpha' = \alpha/[1+P(Y\varepsilon R_2^*)]$.

Again, $P(\theta\varepsilon I) = 1 - \alpha' - p^*$, where

$$p^* = P[\bar{D},\theta\varepsilon I_1] - P[\bar{D},\theta\varepsilon I_2].$$

We use the transformations

$$Y = \frac{2T_{r_1}}{\theta} \text{ and } Z = \frac{2T_{r_3}}{\theta} ,$$

and also define

$$R_1 = [0, \chi^2_{\alpha',2r_1}] ,$$

$$R_2 = \left[\frac{\theta_0 d_1}{\theta} , \frac{\theta_0 d_2}{\theta}\right], \text{ and}$$

$$R_3 = [0, \chi^2_{\alpha',2r_3}].$$

Now we may write

$$p^* = P(Y\varepsilon R_2, Y\varepsilon R_1) - P(Y\varepsilon R_2, Z\varepsilon R_3).$$

As with the two-sided intervals, we can make some general observations about $P(\theta\varepsilon I)$.

i) If

$$\theta < \frac{\theta_0 d_1}{\chi^2_{\alpha',2r_3}} ,$$

then

$$\frac{\theta_0 d_1}{\theta} > \chi^2_{\alpha',2r_3} > \chi^2_{\alpha',2r_1}.$$

Here $P(Y\varepsilon R_2, Y\varepsilon R_1) = P(Y\varepsilon R_2, Z\varepsilon R_3) = 0$, and $P(\theta\varepsilon I) = 1 - \alpha'$.

ii) If

$$\frac{\theta_0 d_1}{\chi^2_{\alpha',2r_3}} \leq \theta \leq \frac{\theta_0 d_1}{\chi^2_{\alpha',2r_1}},$$

then

$$\frac{\theta_0 d_1}{\theta} \geq \chi^2_{\alpha',2r_1}$$

and

$$R_1 \cap R_2 = \emptyset.$$

Since $P(Y\varepsilon R_1, Y\varepsilon R_2) = 0$, $P(\theta\varepsilon I) \geq 1 - \alpha'$.

iii) If

$$\theta \geq \frac{\theta_0 d_2}{\chi^2_{\alpha',2r_1}},$$

then

$$\frac{\theta_0 d_2}{\theta} \leq \chi^2_{\alpha',2r_1}$$

and

$$R_2 \subseteq R_1.$$

Furthermore,

$$\begin{aligned} R(\theta\varepsilon I) &= 1 - \alpha' - P(Y\varepsilon R_2) + P(Y\varepsilon R_2, Z\varepsilon R_3) \\ &= 1 - \alpha' - P(Y\varepsilon R_2)(1-P(Z\varepsilon R_3|Y\varepsilon R_2)) \\ &< 1 - \alpha'. \end{aligned}$$

We can utilize the same approaches to finding an upper bound on p^* that were used in the previous section on two-sided intervals. One bound on p^* is found using the Bonferroni inequality, and the result is again $p^* \leq \alpha'$. Consequently we would set $\alpha' = \frac{\alpha}{2}$ in the intervals (3.1) to ensure that $P(\theta\varepsilon I) \geq 1 - \alpha$ for all θ.

The other approach is to set

$$\Delta = P(Y\varepsilon R_2)P(Z\varepsilon R_3) - P(Y\varepsilon R_2, Z\varepsilon R_3) = 0.$$

Then p^* becomes $P(Y\varepsilon R_2, Y\varepsilon R_1) - (1-\alpha')P(Y\varepsilon R_2)$. By arguments similar to those used in Section II , it can be shown that p^* takes its maximum value when $R_2 \subseteq R_1$. If $R_2 \subseteq R_1$, then $p^* = \alpha' P(Y\varepsilon R_2) \leq \alpha'(1-\alpha')$. Or, we may again define

$$R_2^* = \left[\frac{\theta_0 d_1}{\theta^*} , \frac{\theta_0 d_2}{\theta^*}\right],$$

where θ^* is that value of θ for which $P(Y \varepsilon R_2 | \theta)$ is maximum. So, we obtain $p^* \leq \alpha' P(Y \varepsilon R_2^*)$ when $R_2^* \subseteq R_1$. We arrive at the same terminus as in Section II . We may use $\alpha' = 1 - (1-\alpha)^{1/2}$, which is nearly $\alpha/2$, or $\alpha/[1+P(Y \varepsilon R_2^*)]$ if $R_2^* \subseteq R_1$ and a less conservative one-sided interval is desired.

BIBLIOGRAPHY

Bulgren, W. and Hewett, J. (1973) "Double Sample Tests for Hypotheses about the Mean of an Exponential Distribution." Technometrics, 15, 187-189.

Epstein, B. (1960a) "Estimation from Life Test Data." Technometrics, 2, 447-454.

Epstein, B. (1960b) Statistical Techniques in Life Testing, Office of Technical Services, U.S. Department of Commerce, Washington, D.C.

Epstein, B. and Sobel, M. (1953) "Life Testing I." Journal of the American Statistical Association, 24, 486-502.